Buckling cascades in free sheets

Wavy leaves may not depend only on their genes to make their edges crinkle.

The edge of a torn plastic sheet forms a complex three-dimensional fractal shape. We have found that the shape results from a simple elongation of the sheet in the direction along its edge. Natural growth processes in some leaves, flowers and vesicles could lead to a similar elongation and hence to the generation of characteristic wavy shapes.

We used rectangular plastic sheets pulled from the sides (in the y-direction) to generate a steadily travelling crack (in the x-direction). The high stresses near the crack tip produce an irreversible plastic deformation of the sheet and, as they are relieved, the deformed sheet is free to relax and to adopt a new shape in space.

Surprisingly, the equilibrium shape of the sheet consists of a cascade of waves upon waves along the newly formed edge. The waveform along the edge examined at six levels of magnification is self-similar, with a scaling factor of 3.2 (Fig. 1a). The amplitudes, A, of the waves in the cascades are simply related to their wavelengths λ by $A = 0.15\lambda$ (Fig. 1b); the edge of the sheet is therefore a fractal. The fractal scaling spans 2.5 orders of magnitude and stops at a small length scale that is 6.5 times the sheet's thickness. Measurements of the deformation field revealed an increase in the length of the sheet in the x-direction that depends only on y, and decays smoothly to zero over a distance of a few millimetres from the edge (Fig. 1c).

How can such featureless deformations lead to such complex shapes? The key to this is to realize that the bending rigidity of thin plates is much smaller than their stretching rigidity[1]. Thus, to reduce its elastic energy, the sheet can easily buckle into shapes that remove in-plane compression. The thinner the sheet and the larger the compression, the smaller are the possible wavelengths of the buckles. This principle is seen in action during the crumpling of paper sheets[2–5] and the blistering of thin films[6,7], where sheets adopt very complex shapes in response to a uniform external loading.

Unlike these examples, our sheets (after tearing is complete) are free and not subjected to any external loading. The spontaneous buckling occurs as a consequence of the stresses created by the differential deformation within the sheet. Far from the edge, the deformation is small (Fig. 1c), and the compression leads to buckling that has a single long wavelength. Closer to the edge, the compression increases rapidly, leading to an ordered buckling sequence with smaller and smaller wavelengths.

This phenomenon should be general, and indeed we found buckling cascades in numerical investigations of elastic sheets with deformations similar to those shown in Fig. 1c. Although in the experiment the sheet thinned near the edge (owing to volume conservation), the simulations both with and without thinning revealed buckling cascades. In addition, numerical analysis indicates that the particular scaling properties of the buckling depend on the deformation profile.

A similar increase in length near an edge, and therefore similar internal stresses, can result from growth processes. The wavy structures found in many flower petals, lichens and leaves, such as the beet leaf in Fig. 1d, might be formed in this way. No genetic coding is needed to instruct pieces of a leaf to curl up and curl down. All that is required is a growth process to elongate the sheet along its edge — elasticity takes care of the rest.

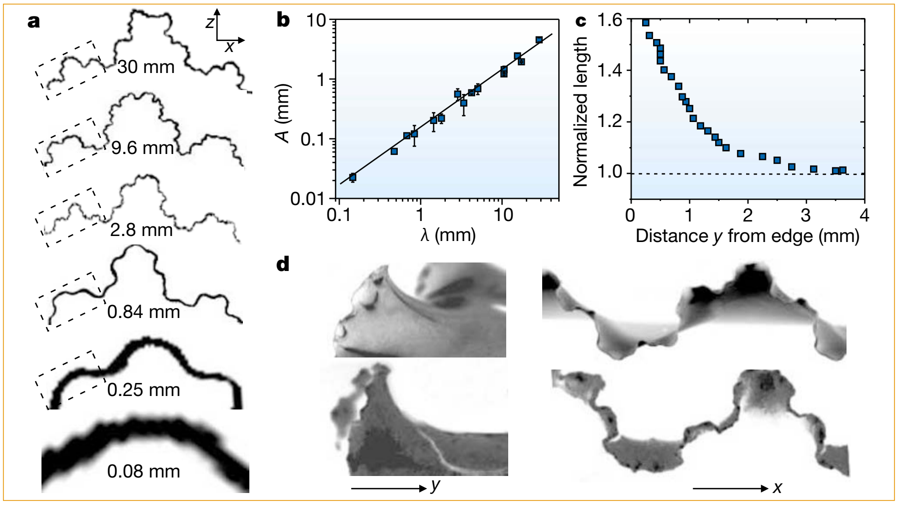

Figure 1 Buckling cascade in a deformed plastic sheet. **a,** Different magnifications of the edge of the sheet (0.012 mm thick). Successive pictures show the dotted boxed region on the left of the previous picture magnified by 3.2; the width of each image is indicated. **b,** The amplitude of a wave in the cascade is related to its wavelength by $A = 0.15\lambda$ (solid line), for sheet thicknesses ranging from 0.012 to 0.5 mm. **c,** The length of a sheet (0.2 mm thick) in the x-direction, normalized by its original length, as a function of distance from the edge. Dashed line indicates the original length. **d,** Side (left) and front (right) views of a plastic sheet (top panels) and a beet leaf (bottom panels), showing the similarity between the two buckled sheets.

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1. Lord Rayleigh *Theory of Sound* Vol. 1, 396 (Dover, New York, 1945).
2. Ben Amar, M. & Pomeau, Y. *Proc. R. Soc. Lond. A* **453,** 729–755 (1997).
3. DiDonna, B. A., Witten, T. A., Venkataramani, S. C. & Kramer, E. M. *Phys. Rev. E* **65,** 016603-1–016603-25 (2002).
4. Matan, K., Williams, R., Witten, T. & Nagel, S. *Phys. Rev. Lett.* **88,** 076101-1–076101-4 (2002).
5. Kramer, E. & Witten, T. *Phys. Rev. Lett.* **78,** 1303–1306 (1997).
6. Argon, A., Gupta, V., Landis, H. & Cornie, J. *J. Mater. Sci.* **24,** 1207–1218 (1989).
7. Gioia, G. & Ortiz, M. *Adv. Appl. Mech.* **33,** 119–192 (1997).

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Thin films

Wrinkling of an elastic sheet under tension

Here we consider the wrinkling of a stretched, slender elastic sheet and derive scaling laws for the wrinkle wavelength and amplitude that are valid far from the onset of buckling. Our results, which can be generalized to stretched and sheared anisotropic and inelastic sheets, could form the basis of a sensitive assay for the mechanical characterization of thin solid films.

When such a thin isotropic elastic sheet of thickness t, width W and length L ($t \ll W < L$), composed of a material with Young's modulus E and Poisson's ratio ν, is subjected to a longitudinal stretching strain, γ, in its plane, it remains flat for $\gamma < \gamma_c$, a critical stretching strain. Further stretching causes the sheet to wrinkle (Fig. 1a).