A History of Mathematical Notations

Volume II

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equality in the 1637 edition of Descartes' *Géométrie* (§ 191). The sign for Aries, placed horizontally, serves with Kästner[1] for "greater than" and "smaller than." The sign for earth is employed extensively in the modern logical exposition of algebra (§ 494). Extensive use of astronomical signs occurs in Leibniz' letters[2] to Jacob Bernoulli; for instance, $\int$ ♃ $\sqrt[\odot]{\;}$ ☽ dx, where each astronomical sign stands for a certain analytic expression (§ 560). Kästner employed the signs for Sun, Moon, Mars, Venus, Jupiter, in the marking of equations, in the place of our modern Roman or Hindu-Arabic numerals.[3] Cauchy sometimes let the sign for Taurus stand for certain algebraic expressions.[4]

THE LETTERS π AND e

395. *Early signs for* 3.1415. . . . ·—John Wallis,[5] in his *Arithmetica infinitorum* (1655), lets the square □ or, in some cases, the Hebrew letter "mem" which closely resembles a square, stand for 4/3,14149 . . . . ; he expresses □ as the ratio of continued products and also, as William Brounker had done before him, in the form of a continued fraction.

Perhaps the earliest use of a single letter to represent the ratio of the length of a circle to its diameter occurs in a work of J. Chr. Sturm,[6] professor at the University of Altdorf in Bavaria, who in 1689 used the letter e in a statement, "si diameter alicuius circuli ponatur a, circumferentiam appellari posse ea (quaecumque enim inter eas fuerit ratio, illius nomen potest designari littera e)." Sturm's letter failed of general adoption.

Before Sturm the ratio of the length of a circle to its diameter was represented in the fractional form by the use of two letters. Thus,

[1] A. G. Kästner, *Anfangsgründe der Arithmetik, Geometrie* . . . . (Göttingen, 1758), p. 89, 385.

[2] C. I. Gerhardt, *Leibnizens Mathematische Schriften*, Vol. III (Halle, 1855), p. 100.

[3] A. G. Kästner, *Anfangsgründe der Analysis endlicher Grössen* (Göttingen, 1760), p. 55, 269, 336, 358, 414, 417, and other places.

[4] A. L. Cauchy, *Comptes rendus*, Vol. XXIV (1847); *Œuvres complètes* (1st ser.), Vol. X, p. 282.

[5] John Wallis, *Arithmetica infinitorum* (Oxford, 1655), p. 175, 179, 182.

[6] J. Christoph Sturm, *Mathesis enucleata* (Nürnberg, 1689), p. 81. This reference is taken from A. Krazer's note in *Euleri opera omnia* (1st ser.), Vol. VIII, p. 134.

William Oughtred[1] designated the ratio (§ 185) by $\frac{\pi}{\delta}$. He does not define π and δ separately, but no doubt π stood for periphery and δ for diameter. The radius he represents by R. We quote from page 66 of the 1652 edition: "Si in circulo sit $7.22::\delta.\pi::113.355$: erit $\delta.\pi::2R.P$: periph. Et $\pi.\delta::\frac{1}{2}P.R$: semidiam. $\delta.\pi::Rq$. Circul. Et $\pi.\delta::\frac{1}{4}Pq$. Circul." Oughtred's notation was adopted by Isaac Barrow[2] and by David Gregory.[3] John Wallis[4] in 1685 represented by π the "periphery" described by the center of gravity in a revolution. In 1698 De Moivre[5] designated the ratio of the length of the circle to the *radius* by $\frac{c}{r}$.

396. *First occurrence of the sign* π.—The modern notation for 3.14159 . . . . was introduced in 1706. It was in that year that William Jones[6] made himself noted, without being aware that he was doing anything noteworthy, through his designation of the ratio of the length of the circle to its diameter by the letter π. He took this step without ostentation. No lengthy introduction prepares the reader for the bringing upon the stage of mathematical history this distinguished visitor from the field of Greek letters. It simply came, unheralded, in the following prosaic statement (p. 263):

"There are various other ways of finding the *Lengths* or *Areas* of particular *Curve Lines*, or *Planes*, which may very much facilitate the Practice; as for instance, in the *Circle*, the Diameter is to the Circumference as 1 to $\overline{\frac{16}{5}-\frac{4}{239}}-\frac{1}{3}\overline{\frac{16}{5^3}-\frac{4}{239^3}}-$, &c.$=3.14159$, &c.$=\pi$. This *series* (among others for the same purpose, and drawn from the same Principle) I received from the Excellent Analyst, and my much esteem'd Friend Mr. *John Machin;* and by means thereof, *Van*

[1] W. Oughtred, *Clavis mathematicae* (1652), p. 66. This symbolism is given in the editions of this book of 1647, 1648, 1652, 1667, 1693, 1694. It is used also in the Appendix to the *Clavis*, on "Archimedis de Sphaera et Cylindro declaratio." This Appendix appeared in the editions of 1652, 1667, 1693.

[2] W. Whewell, *The Mathematical Works of Isaac Barrow* (Cambridge, 1860), p. 380, Lecture XXIV.

[3] David Gregory, *Philosophical Transactions*, Vol. XIX (London, 1697), p. 652 except that he writes $\frac{\pi}{\rho}$, ρ being the radius.

[4] John Wallis, *Treatise of Algebra* (1685), "Additions and Emendations," p. 170.

[5] De Moivre, *Philosophical Transactions*, Vol. XIX (1698), p. 56.

[6] William Jones, *Synopsis palmariorum matheseos* (London, 1706), p. 263.

Ceulen's Number, or that in Art. 64.38. may be Examin'd with all desirable Ease and Dispatch." Then he writes "$d=c\div\pi$" and "$c=d\times\pi$."

This was not the first appearance of the letter π in Jones's book of 1706. But in earlier passages the meanings were different. On page 241 it was used in lettering a geometric figure where it represented a point. On page 243 one finds "Periphery (π)," as previously found in Wallis.

Nor did the appearance of $\pi=3.14159\ldots\ldots$ on the stage attract general attention. Many mathematicians continued in the old way. In 1721 P. Varignon[1] wrote the ratio $\delta.\pi$, using for ratio the dot of Oughtred.

397. *Euler's use of* π.—In 1734 Euler[2] employed p instead of π and g instead of $\frac{\pi}{2}$. In a letter of April 16, 1738, from Stirling to Euler, as well as in Euler's reply, the letter p is used.[3] But in 1736 he[4] designated that ratio by the sign $1:\pi$ and thus either consciously adopted the notation of Jones or independently fell upon it. Euler says: "Si enim est $m=\frac{1}{2}$ terminus respondens inuenitur $\frac{\pi}{2}$ denotante $1:\pi$ rationem diametri ad peripheriam." But the letter is not restricted to this use in his *Mechanica*, and the definition of π is repeated when it is taken for 3.1415. . . . . He represented 3.1415 . . . . again by π in 1737[5] (in a paper printed in 1744), in 1743,[6] in 1746,[7] and in 1748.[8] Euler and Goldbach used $\pi=3.1415\ldots\ldots$ repeatedly in their correspondence in 1739. Johann Bernoulli used in 1739, in his correspondence with Euler, the letter c (*circumferentia*), but in a letter of 1740

[1] Pierre Varignon, *Histoire de l'Académie r. des sciences*, année 1721 (Paris, 1723), *Mémoires*, p. 48.

[2] Euler in "De summis serierum reciprocarum," *Comm. Acad. Petr.*, Vol. VII (1734–35), p. 123 ff. See von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Vol. II (Leipzig, 1903), p. 110.

[3] Charles Tweedie's *James Stirling* (Oxford, 1922), p. 179, 180, 185, 188.

[4] L. Euler, *Mechanica sive motus scientia analytice exposita*, Vol. I (Petrograd, 1736), p. 119, 123; Vol. II, p. 70, 80.

[5] L. Euler in *Comm. Acad. Petr. ad annum 1737*, IX (1744), p. 165. See A. von Braunmühl, *op. cit.*, Vol. II, p. 110. Euler says: "Posito π pro peripheria circuli, cuius diameter est 1, ... "

[6] L. Euler in *Miscellanea Berolinensia*, Vol. VII (1743), p. 10, 91, 136.

[7] L. Euler in *Histoire de l'académie r. des sciences, et de belles lettres*, année 1745 (Berlin, 1746), p. 44.

[8] L. Euler, *op. cit.*, année 1748 (Berlin, 1750), p. 84.

he began to use π. Likewise, Nikolaus Bernoulli employed π in his letters to Euler of 1742.[1] Particularly favorable for wider adoption was the appearance of π for 3.1415 . . . . in Euler's *Introductio in analysin infinitorum* (1748). In most of his later publications, Euler clung to π as his designation of 3.1415. . . . .

398. *Spread of Jones's notation.*—In 1741, $\pi=3.14159$ . . . . is used in Sherwin's *Tables.*[2] Nevertheless, mathematicians in general were slow in following suit. In 1748 Diderot[3] wrote, "Soit le rapport du diametre à la circumférence $=\frac{1}{c}$ ... ." J. A. Segner varied in his practice; in 1751[4] he let π stand for the ratio, but in 1767 he[5] represented 3.14159 . . . . by $\delta:\pi$, as did Oughtred more than a century earlier. Says Segner: "Si ratio diametri ad peripheriam circuli, quam dedimus, vel alia verae satis propinqua, $\delta:\pi$, et sit diameter circuli data d, erit eiusdem circuli peripheria$=\frac{\pi}{\delta}.d$." Again, later, he lets π be "dimidium peripheriae" of the circle.[6] Even more vacillating was Kästner, who in his *Anfangsgründe* of 1758 lets $1:P$ stand for the ratio of diameter to circumference,[7] and π for the circumference. He uses P in this sense in his plane geometry, and the early part of his solid geometry. Then all of a sudden he writes (p. 323) the ratio in the form $1:\pi$ and continues this notation over nine consecutive pages. Further on (p. 353) in his trigonometry he puts $\cos u=\pi$ and $\sin u=p$; he writes, on page 367, $\cos A=\pi$, and on page 389, $\cos AP=\pi$. It cannot be said that in 1758 Kästner had settled upon any one fixed use of the letter π. In 1760 his practice had not changed;[8] he lets π be coefficient of the $(n+1)$th term of an equation; later he puts π equal to the algebraic irrational $\sqrt{a}$,then $\pi=\sqrt[a]{-1}$, then π is an angle APM

[1] See Paul H. von Fuss, *Correspondance mathématique et physique de quelques célèbres géomètres du XVIII siècle* (1843). Also F. Rudio, *Archimedes, Huygens, Lambert, Legendre* (Leipzig, 1892), p. 53.

[2] H. Sherwin's *Mathematical Tables* (3d éd.; revised by William Gardiner, London, 1741), p. 44.

[3] Denys Diderot, *Mémoires sur différens sujets de mathematiques* (Paris, 1748), p. 27.

[4] J. A. Segner, *Histoire de l'académie*, année 1751 (Berlin, 1753), p. 271.

[5] J. A. de Segner, *Cursus Mathematici*, Pars I (2d ed.; Halae, 1767), p. 309.

[6] Segner, *op. cit.*, Pars IV (Halae, 1763), p. 3.

[7] A. G. Kästner, *Anfangsgründe der Arithmetik, Geometrie und Trigonometrie* (Göttingen, 1758), p. 267, 268.

[8] A. G. Kästner, *Anfangsgründe der Analysis endlicher Grössen* (Göttingen, 1760), p. 107, 117, 211, 228, 254, 296, 326, 327, 413, 432.

and $\pi=R$, then $\pi=\tan MpH$, then π is a coefficient in the cubic $z^3+\pi z+p=0$. After that $\pi=3.14159\ldots.$, then π is a general exponent of the variable x, and is $=0$ in a particular case and $=3$ in another, then again π is the coefficient of a term in an equation, and an exponent of x. Evidently π was still serving him in the rôle of a general-utility symbol. But in 1771, at last, Kästner[1] regularly reserved $\pi=3.14159\ldots..$

Nicolas de Beguelin[2] in 1751 adopted the notation $\pi=3.14159\ldots.$ as did also Daniel Bernoulli[3] in 1753, G. W. Krafft[4] in 1753, Daviet de Foncenex[5] in 1759.

Another noted German writer of textbooks of the eighteenth century, W. J. G. Karsten, uses π in the first volume of his *Lehrbegrif*[6] to represent a polygon, and uses no letter for 3.14 . . . . . But in the second volume he is definite: "Wenn man hinführo ein für allemahl die Zahl 3, 1415926 u.s.f. $=\pi$ setzt, ... so ist $p=2r\pi=\pi d$." One finds π for 3.14159 . . . . in publications of C. A. Vandermonde[7] in 1770, and Laplace[8] in 1782. About the middle of the eighteenth century the letter π was used frequently by French mathematicians in mechanics and astronomy for other designations than 3.141 . . . . , but in the latter part of that century 3.141 . . . . came to be generally designated by π. Unusual is the procedure of Wessel,[9] who writes $\pi=360°$, and of L. N. M. Carnot, who, in his *Géométrie de position* (1803), page 138, takes the radius to be unity and *a-fourth* of the length of the circle to be π, so that "$\sin(\pi\pm a)=+\cos a$." Another unusual procedure is that of D. Lardner,[10] who lets π be the "approxi-

[1] A. G. Kästner, *Dissertationes mathematicae et physicae* (Altenbvrgi, 1771), p. 41, 66, 67.

[2] Beguelin, *op. cit.*, année 1751 (Berlin, 1753), p. 444.

[3] Daniel Bernoulli, *Histoire de l'académie*, année 1753 (Berlin, 1755), p. 156.

[4] Georg Wolffgang Krafft, *Institutiones Geometriae Sublimioris* (Tübingen, 1753), p. 122.

[5] Daviet de Foncenex in *Miscellanea philosophico-mathematica Taurinensis*, Vol. I (1759), p. 130.

[6] W. J. G. Karsten, *Lehrbegrif der gesamten Mathematik. 1. Theil* (Greifswald, 1767), p. 304, 412.

[7] Vandermonde in *Histoire de l'Académie des Sciences*, année 1770 (Paris, 1773), p. 494.

[8] Laplace in *op. cit.*, année 1782 (Paris, 1785), p. 15.

[9] Caspar Wessel, *Essai sur la représentation analytique de la direction* (Copenhague, 1897), p. 15. This edition is a translation from the Danish (1799).

[10] Dionysius Lardner, *The First Six Books of the Elements of Euclid* (London, 1828), p. 278.

mate ratio of the circumference of a circle to its diameter," but does not state which approximate value it represents. The Italian, Pietro Ferroni,[1] in 1782 wrote the capital letters P for 3.14159 . . and Π for 6.283 . . Perhaps the earliest elementary French schoolbook to contain π in regular use was A. M. Legendre's *Éléments de géométrie* (1794), page 121.

399. *Signs for the base of natural logarithms.*—The need of a symbol to represent the base of the natural system of logarithms presented itself early in the development of the calculus. Leibniz[2] used the letter b in letters to Huygens of October 3/13, 1690, and January 27, 1691. In the latter he considers $t=\int\frac{dv}{1-v^2}$ and writes $b^t_{:}=\frac{1+v}{1-v}$, "b estant une grandeur constante, dont le logarithme est 1, et le logarithme de 1 estant 0." A reviewer[3] of G. Cheyne's *Fluxionum methodus inversa* writes in 1703, "$\int dx:x=lx$ et $X^x_{:}=a^y$. (seu cum $la=1$)$x\ lx=y$," thus suggesting the letter a.

400. *The letter e.*—The introduction of the letter e to represent the base of the natural system of logarithms is due to L. Euler. According to G. Eneström, it occurs in a manuscript written in 1727 or 1728, but which was not published until 1862.[4] Euler used e again in 1736 in his *Mechanica*,[5] Volume I, page 68, and in other places, as well as in articles of the years[6] 1747 and 1751. Daniel Bernoulli[7] used e in this sense in 1760, J. A. Segner[8] in 1763, Condorcet[9] in 1771, Lambert[10] in

[1] Pietro Ferroni, *Magnitudinum exponentialium . . . . theoria Florence* (1782), p. 228, 252.

[2] C. I. Gerhardt, *Leibnizens Mathematische Schriften*, Vol. II (Berlin, 1850), p. 53, 76.

[3] *Acta eruditorum* (Leipzig, 1703), p. 451.

[4] Euler's art., "Meditatio in experimenta explosione tormentorum nuper instituta," in the *Opera posthuma* (1862), Vol. II, p. 800–804. See G. Eneström, *Bibliotheca mathematica* (3d ser.), Vol. XIV (1913–14), p. 81.

[5] L. Euler, *Mechanica sive motus scientia analytice exposita* (St. Petersburg, 1736), p. 251, 256; also in *Comm. Acad. Petr.*, Vol. VII (1740) p. 146.

[6] L. Euler in *Histoire de l'Academie r. d. sciences et d. belles lettres de Berlin*, année 1745 (Berlin, 1746), p. 185; année 1751 (Berlin, 1753), p. 270.

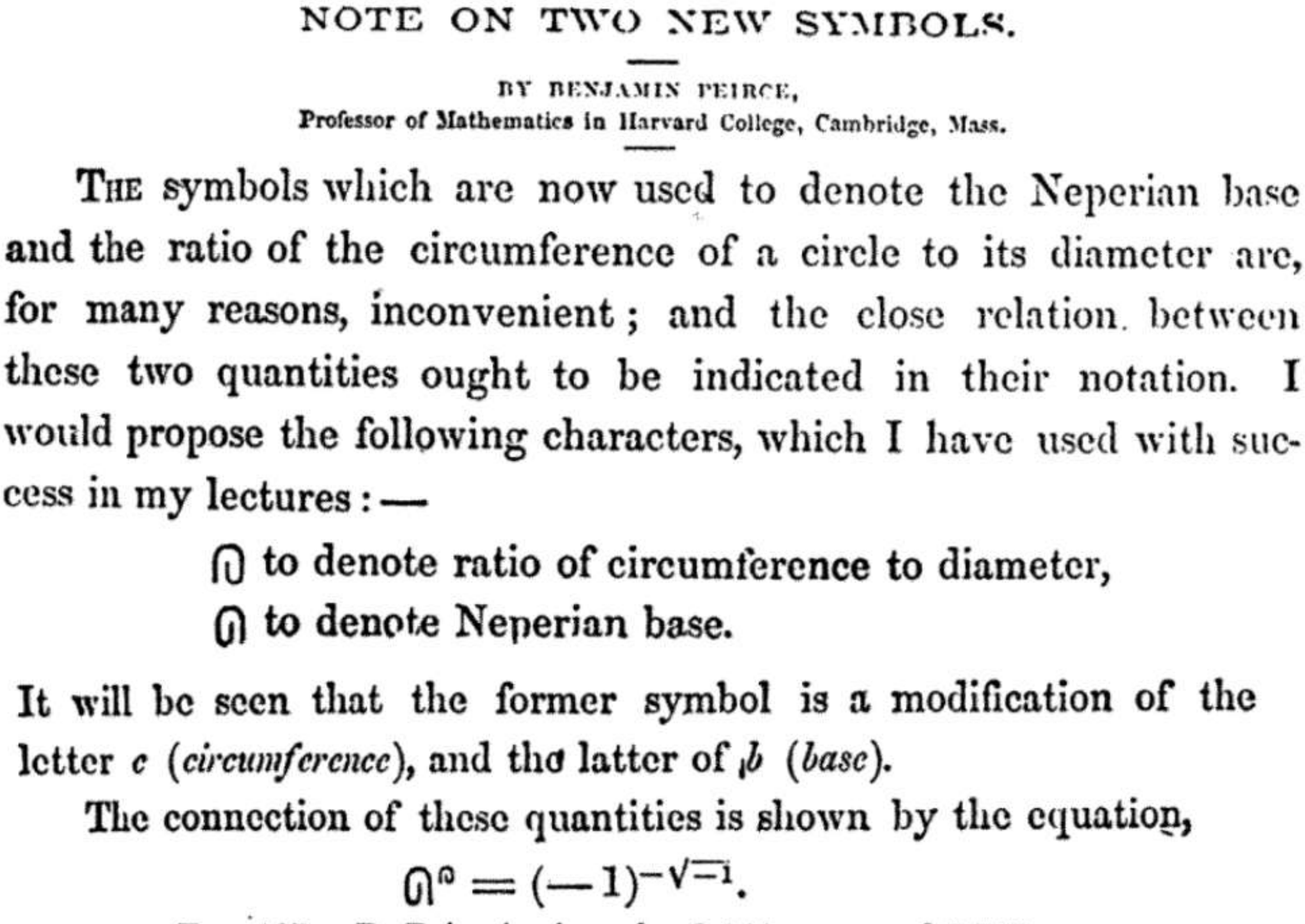
[7] Daniel Bernoulli in *Histoire de l'Académie r. d. sciences*, année 1760 (Paris, 1766), p. 12.

[8] J. A. Segner, *Cursus mathematici*, Paris IV (Halae, 1763), p. 60.

[9] N. C. de Condorcet, *Histoire de l'académie*, année 1771 (Paris, 1774), p. 283.

[10] J. H. Lambert in *Histoire de l'Académie r. d. sciences et d. belles lettres*, année 1764 (Berlin, 1766), p. 188; année 1764 (Berlin, 1766), p. 412.

1764, J. A. Fas[1] in 1775. On the other hand, D'Alembert[2] in 1747 and in 1764 used the letter c for 2.718 . . . . , as did also the astronomer Daniel Melandri[3] of Upsala in 1787. The letter e for 2.718 is found in Abbé Sauri,[4] in E. Bézout,[5] in C. Kramp.[6] In Italy, P. Frisi,[7]



NOTE ON TWO NEW SYMBOLS.

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THE symbols which are now used to denote the Neperian base and the ratio of the circumference of a circle to its diameter are, for many reasons, inconvenient; and the close relation between these two quantities ought to be indicated in their notation. I would propose the following characters, which I have used with success in my lectures:—

to denote ratio of circumference to diameter,

to denote Neperian base.

It will be seen that the former symbol is a modification of the letter *c* (*circumference*), and the latter of *b* (*base*).

The connection of these quantities is shown by the equation,

$$= (-1)^{-\sqrt{-1}}.$$

FIG. 107.—B. Peirce's signs for 3.141 . . . . and 2.718 . . . .

in 1782, and Pietro Ferroni,[8] in the same year, used C for 2.718 . . . . , but Paoli[9] adopted the e. A. de Morgan[10] in 1842 used the epsilon ϵ for 2.718 . . . . and E for $e^{\sqrt{-1}}$.

[1] J. A. Fas, *Inleiding tot de Kennisse en het gebruyk der Oneindig Kleinen* (Leyden, 1775), p. 71.

[2] D'Alembert in *Histoire de l'académie*, année 1747 (Berlin, 1748), p. 228; année 1764 (Berlin, 1766), p. 412.

[3] Daniel Melandri in *Nova Acta Helvetica physico-mathematica*, Vol. I (Basel, 1787), p. 102.

[4] L'Abbé Sauri, *Cours de mathématiques*, Tome III (Paris, 1774), p. 35.

[5] E. Bézout, *Cours de mathématiques*, Tome I (2d ed.; Paris, 1797), p. 124.

[6] C. Kramp, *Eléments d'arithmétique* (Cologne, 1808), p. 28.

[7] Paulii Frisii, *Operum tomus primus* (Mediolani, 1782), p. 195.

[8] Pietro Ferroni, *Magnitudinum exponentialium logarithmorum et Trigonometriae sublimis theoria* (Florence, 1782), p. 64.

[9] Pietro Paoli, *Elementi d'algebra*, Tomo I (Pisa, 1794), p. 216.